

Entropic uncertainty relations and the stabilizer formalism

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Entropic uncertainty relations express the quantum mechanical uncertainty principle by quantifying uncertainty in terms of entropy. Central questions include the derivation of lower bounds on the total uncertainty for given observables, the characterization of observables that allow strong uncertainty relations, and the construction of such relations for the case of several observables. We demonstrate how the stabilizer formalism can be applied to these questions. We show that the Maassen–Uffink entropic uncertainty relation is tight for the measurement in any pair of stabilizer bases. We compare the relative strengths of variance-based and various entropic uncertainty relations for dichotomic anticommuting observables.

I. INTRODUCTION

In quantum mechanics, one cannot predict a measurement outcome with certainty unless the system is in an eigenstate of the observable being measured. It follows that if two or more observables have no common eigenstate, it is not possible to prepare the system such that for each observable only one measurement outcome can occur. This is known as the uncertainty principle,^{1–4} which is quantitatively formulated in terms of uncertainty relations.

Uncertainty relations not only describe a fundamental quantum mechanical concept, but have also found application, e. g., in quantum cryptography^{5–7} and entanglement detection.^{8–10} Entropic uncertainty relations^{11–15} have turned out to be particularly useful. More recently, the uncertainty principle has also been formulated in terms of majorization relations.¹⁶ In Ref. 17, Berta *et al.* derived an entropic uncertainty relation for a system which is entangled to a quantum memory. Access to this memory can then be used to lower the uncertainties of measurements on the system. This relation has been the subject of recent experiments.^{18,19}

Historically, the first and still the most celebrated uncertainty relation was given by Heisenberg^{1,2} and applies to canonically conjugate observables such as position and momentum, stating that $\Delta^2(q)\Delta^2(p) \geq \hbar^2/4$. It was generalized to arbitrary observables in the form of Robertson’s³ uncertainty relation $\Delta^2(A)\Delta^2(B) \geq |\langle[A, B]\rangle|^2/4$. On closer inspection, the latter does not have all desirable properties of an uncertainty relation, in particular since it is trivial for any eigenstate of either observable. Robertson’s relation has also been generalized to the case of more than two observables.^{20,21}

A different approach is based on the idea of quantifying uncertainty by the entropy of the probability distribution for the measurement outcomes.^{11–15} As a consequence, the resulting uncertainty relations depend only on the eigenstates, but not on the eigenvalues of the observables. That is, they are independent of the labelling of the measurement results, which in finite dimensions is essentially

arbitrary. For this reason, entropic uncertainty relations can be regarded as a more natural formulation of the uncertainty principle, at least in the case of finitely many measurement outcomes, which we consider here. For a review of entropic uncertainty relations see, e. g., Ref. 15.

An entropic uncertainty relation [see Eq. (6) for an example] gives a lower bound on the sum of the entropies for the measurement outcomes of some observables on a quantum state. This bound necessarily depends on the observables, but preferably is independent of the state. For the case of two observables, such bounds have been determined, e. g., in Refs. 12–14. Finding the best bound is in general not easy. Putting differently, the question here is a characterization of observables which admit strong uncertainty relations. So far, most results apply to the case of two observables only,^{15,22} despite the generalization of the theory to several observables being an interesting problem.

In this article, we demonstrate how the stabilizer formalism²³ can be applied to these questions. This formalism provides an efficient description of certain many-qubit states, including highly entangled ones. In the field of quantum information theory, stabilizer states and the more special case of graph states are widely used. For example, a particular class of graph states, namely, cluster states, serve as a universal resource for measurement-based quantum computation.^{24,25} Cluster states are also interesting because they are particularly robust against decoherence, independently of the system size.^{26,27} Graph states have been employed as codewords of error-correcting codes.²⁸ Finally, the stabilizer formalism has been used for entanglement detection.²⁹

Using the stabilizer formalism, we first investigate the question for which observables the Maassen–Uffink uncertainty relation [see Eq. (6)] is tight. We show that this is the case for the measurement in any two stabilizer bases. We then turn our attention to the many-observable setting, focussing on dichotomic anticommuting observables. Generalizing a result by Wehner and Winter,²² we provide a systematic construction of uncertainty relations. The family of uncertainty relations we obtain con-

tains both entropic and variance-based ones. We compare the relative strengths of these relations. Finally, we apply them to the stabilizing operators of two stabilizer states.

This article is organized as follows: In Sec. II, we introduce our notation and review previous results. This includes a short introduction to stabilizers and graph states. Section III contains our results on measurements in stabilizer bases, and Sec. IV those on several dichotomic anticommuting observables. In Sec. V, the latter are applied to the stabilizing operators of two stabilizer states. Finally, Sec. VI is devoted to a discussion of the results.

II. STATEMENT OF THE PROBLEM

A. Entropic uncertainty relations

We consider the following general situation: when measuring an observable A on a state ρ , the measurement outcome a_i occurs with probability $p_i = \text{Tr}(\Pi_i \rho)$, where $A = \sum_{i=1}^m a_i \Pi_i$ is the spectral decomposition of the observable and the a_i are mutually distinct. For an entropy function S , we denote by $S(A|\rho)$ the entropy of this probability distribution $P = (p_1, \dots, p_m)$. In this article, we will use the Shannon entropy

$$S^S(P) = -\sum_{i=1}^m p_i \log(p_i) \quad (1)$$

as well as the min-entropy

$$S^{\min}(P) = -\log(\max_i p_i) \quad (2)$$

and the Tsallis entropy

$$S_q^T(P) = \frac{1 - \sum_{i=1}^m (p_i)^q}{q - 1}, \quad q > 1. \quad (3)$$

In the limit $q \rightarrow 1$, the Tsallis entropy gives the Shannon entropy. We use the logarithm to base 2 throughout.

We are interested in uncertainty relations for a family of observables $\{A_1, \dots, A_L\}$ of the form

$$\frac{1}{L} \sum_{k=1}^L S(A_k|\rho) \geq c_{\{A_k\}}, \quad (4)$$

where S is an entropy function. The lower bound $c_{\{A_k\}}$ may depend on the observables, but shall be independent of the state. For a given set of observables, an uncertainty relation is called tight, if a state ρ_0 exists that attains the lower bound, $1/L \sum_{k=1}^L S(A_k|\rho_0) = c_{\{A_k\}}$.

Clearly the entropy of an observable depends only on its eigenstates and is independent of its eigenvalues, as long as they are nondegenerate. We will therefore not distinguish between a nondegenerate observable A and its eigenbasis \mathcal{A} . The Shannon entropy satisfies $0 \leq S^S(\mathcal{A}|\rho) \leq \log(m)$, where m is the length of the

basis \mathcal{A} . If we choose for ρ one of the basis states of \mathcal{A}_k , we have $1/L \sum_{k=1}^L S(\mathcal{A}_k|\rho) \leq \log(m)(L-1)/L$, because in this case the entropy is zero for one basis and upper bounded by $\log(m)$ for the remaining $L-1$ bases. This implies that for the Shannon entropy the right-hand side of Eq. (4) cannot exceed $\log(m)(L-1)/L$. An uncertainty relation that reaches this limit is called maximally strong, and the corresponding measurements are called maximally incompatible.¹⁵ In other words, maximal incompatibility means that if the outcome of one measurement is certain, the outcomes of the remaining measurements are completely random.

A related notion is mutual unbiasedness. Two orthonormal bases $|a_i\rangle$ and $|b_i\rangle$, $i = 1, \dots, d$, are called mutually unbiased if

$$|\langle a_i | b_j \rangle| = \frac{1}{\sqrt{d}}, \quad (5)$$

for all i and j . Pairwise mutual unbiasedness is a necessary, but for more than two bases not a sufficient condition for the existence of a maximally strong uncertainty relation.¹⁵

For the Shannon entropies of $L = 2$ measurement bases, Maassen and Uffink¹⁴ have proven the following result:

Maassen–Uffink uncertainty relation. *For any two measurement bases $\mathcal{A} = \{|a_i\rangle\}$ and $\mathcal{B} = \{|b_i\rangle\}$,*

$$\frac{1}{2}[S^S(\mathcal{A}|\rho) + S^S(\mathcal{B}|\rho)] \geq -\log(\max_{i,j} |\langle a_i | b_j \rangle|). \quad (6)$$

In the case of mutually unbiased bases the Maassen–Uffink relation is maximally strong and thus tight, equality holding for any of the basis states. (Note that for two arbitrary observables the entropy sum is in general not minimized by an eigenstate of either of them.³⁰)

B. The stabilizer formalism and graph states

As our main tool, we will employ the stabilizer formalism. This formalism allows to describe certain many-qubit states, among them graph states, in an efficient manner. For a review of this topic see, e. g., Ref. 23.

The n -qubit Pauli group consists of all tensor products of n Pauli matrices, including the identity, with prefactors ± 1 and $\pm i$. Any commutative subgroup of the Pauli group that has 2^n elements and does not contain $-\mathbb{1}$ has a unique common eigenstate with eigenvalue $+1$. This state is called stabilizer state; the group is called the stabilizer group and the group elements are called the stabilizing operators of the state. The stabilizer group defines in fact a complete basis of common eigenstates, the stabilizer state being one of them. We refer to this basis as stabilizer basis. Any basis state is again a stabilizer state, whose stabilizer group is obtained from the original one by flipping the signs of some of its elements.

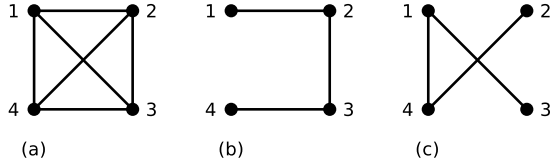


FIG. 1. Graphs of the graph states discussed in the text.

The most important examples of stabilizer states are graph states. In fact, it can be shown that any stabilizer state is equivalent to a graph state under a local unitary operation (more specifically, a local Clifford operation).²³ A graph state is described by a simple undirected graph, whose vertices represent qubits and whose edges represent the interactions that have created the graph state from a product state [see Fig. 1 for examples]. More precisely, with the completely disconnected n -vertex graph, we associate the state $|+\rangle^{\otimes n}$, where $|+\rangle$ is the eigenvector of σ_x with eigenvalue $+1$. An edge between two vertices stands for the controlled phase gate applied to the corresponding pair of qubits, which in the standard σ_z -basis is $C = \text{diag}(1, 1, 1, -1)$. The stabilizing operators of a graph state can immediately be read off the graph: with qubit i , we associate the operator

$$K_i = \sigma_x^{(i)} \prod_{j \in N(i)} \sigma_z^{(j)}, \quad (7)$$

the product being over the neighbourhood $N(i)$ of vertex i , that is, all vertices directly connected to it by an edge. Here $\sigma_x^{(i)}$ and $\sigma_z^{(i)}$ denote the Pauli matrices σ_x and σ_z acting on qubit i . The operators K_i generate the stabilizer group. Any state of the stabilizer basis (here called graph state basis) can be obtained from the graph state by applying the operation σ_z on a subset of the qubits.

As an example, let us consider the graphs in Fig. 1 (a) and (b). The generators of the stabilizer group of the first state are $K_1 = \sigma_x \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z$, $K_2 = \sigma_z \otimes \sigma_x \otimes \sigma_z \otimes \sigma_z$, $K_3 = \sigma_z \otimes \sigma_z \otimes \sigma_x \otimes \sigma_z$, and $K_4 = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_x$; for the second state, they are $K_1 = \sigma_x \otimes \sigma_z \otimes \mathbb{1} \otimes \mathbb{1}$, $K_2 = \sigma_z \otimes \sigma_x \otimes \sigma_z \otimes \mathbb{1}$, $K_3 = \mathbb{1} \otimes \sigma_z \otimes \sigma_x \otimes \sigma_z$, and $K_4 = \mathbb{1} \otimes \mathbb{1} \otimes \sigma_z \otimes \sigma_x$. Under local unitary operations, the corresponding graph states are equivalent to the 4-qubit Greenberger–Horne–Zeilinger (GHZ) state $|\text{GHZ}_4\rangle = (|0000\rangle + |1111\rangle)/\sqrt{2}$ and the 4-qubit linear cluster state $|C_4\rangle = (|0000\rangle + |0011\rangle + |1100\rangle - |1111\rangle)/2$, respectively.

III. A STRONG UNCERTAINTY RELATION FOR STABILIZER BASES

In this section, we investigate the conditions under which the Maassen–Uffink uncertainty relation is tight. We show that this is the case for the measurement in any two stabilizer bases.

We shall need the following lemma:

Lemma 1. *If a pair of bases $\mathcal{A} = \{|a_i\rangle\}$ and $\mathcal{B} = \{|b_i\rangle\}$ satisfies*

$$|\langle a_i | b_j \rangle| \in \{0, r\} \quad \forall i, j \quad (8)$$

for some r , then the Maassen–Uffink relation Eq. (6) for the measurement in these bases is tight. Equality occurs for any of the basis states.

Proof. For $\rho = |b_{j_0}\rangle\langle b_{j_0}|$ the Maassen–Uffink relation reads

$$-\sum_i p_i \log(p_i) \geq -\log(\max_i p_i) \quad \text{where } p_i = |\langle a_i | b_{j_0} \rangle|^2. \quad (9)$$

Note that the right-hand side is the min-entropy of the probability distribution p_i . By assumption $p_i \in \{0, r^2\}$ for all i . It follows that equality holds. \square

The main result of this section is the following theorem:

Theorem 2. *For the measurement in a pair of stabilizer bases, the Maassen–Uffink uncertainty relation Eq. (6) is tight. The bound is attained by any of the basis states.*

The proof is based on a result on mutually unbiased bases, which is due to Bandyopadhyay *et al.* (see the proof of Theorem 3.2 in Ref. 31):

Theorem 3 (Ref. 31). *Let C_1 and C_2 each be a set of d commuting unitary $d \times d$ -matrices. Furthermore assume that $C_1 \cap C_2 = \{\mathbb{1}\}$ and that all matrices in $C_1 \cup C_2$ are pairwise orthogonal with respect to the Hilbert–Schmidt scalar product. Then the eigenbases defined by either set of matrices are mutually unbiased.*

Proof of Theorem 2. Throughout the proof we consider two stabilizing operators that differ only by a minus sign as equal. Let S_1 and S_2 be the two stabilizer groups and define $C_0 = S_1 \cap S_2$. Then C_0 is a subgroup of both S_1 and S_2 . We consider the factor groups $C_1 = S_1/C_0$ and $C_2 = S_2/C_0$. The groups C_0 , C_1 , and C_2 are all stabilizer groups, though the spaces stabilized by them are in general not one-dimensional. This gives us a decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_{12}$, where C_0 acts trivially on \mathcal{H}_{12} and C_1 and C_2 act trivially on \mathcal{H}_0 . By the previous theorem, C_1 and C_2 define mutually unbiased bases $|c_i^{(1)}\rangle$ and $|c_i^{(2)}\rangle$ of \mathcal{H}_{12} . It follows that the stabilizer bases can be written as $|s_{ij}^{(1)}\rangle = |c_i^{(0)}\rangle \otimes |c_j^{(1)}\rangle$ and $|s_{ij}^{(2)}\rangle = |c_i^{(0)}\rangle \otimes |c_j^{(2)}\rangle$, respectively, where $|c_i^{(0)}\rangle$ is the basis of \mathcal{H}_0 defined by C_0 . The stabilizer bases thus satisfy the condition of Lemma 1 with $r = (\dim \mathcal{H}_{12})^{-1/2}$. \square

In Appendix A, we give an alternative proof that does not require the result on mutually unbiased bases. In Appendix B, we develop a method to calculate the right-hand side of the uncertainty relation explicitly for certain classes of graph states.

IV. UNCERTAINTY RELATIONS FOR SEVERAL DICHOTOMIC ANTICOMMUTING OBSERVABLES

Little is known about uncertainty relations for more than two measurements¹⁵ (see, however, Ref. 32). Following Wehner and Winter,²² we will concentrate on dichotomic anticommuting observables. An observable is called dichotomic if it has exactly two distinct eigenvalues. We will always normalize dichotomic observables such that their eigenvalues are ± 1 . In other words, these observables square to the identity.

The following result has been called a meta-uncertainty relation,^{15,22} for reasons that soon will become apparent.

Lemma 4. *Let A_1, \dots, A_L be observables which anticommute pairwise $\{A_k, A_\ell\} = 0$ for $k \neq \ell$ and which have eigenvalues ± 1 . Then $\sum_{k=1}^L \langle A_k \rangle^2 \leq 1$, or equivalently,*

$$\sum_{k=1}^L \Delta^2(A_k) \geq L - 1, \quad (10)$$

where $\Delta^2(A) = \langle (A - \langle A \rangle)^2 \rangle$ is the variance of A .

The following proof of this lemma was given in Ref. 29. For an alternative proof, based on the Clifford algebra, see Ref. 22.

Proof. Choose real coefficients $\lambda_1, \dots, \lambda_L$ with $\sum_{k=1}^L \lambda_k^2 = 1$. Because of anticommutativity and $A_k^2 = \mathbb{1}$, we have $(\sum_{k=1}^L \lambda_k A_k)^2 = \sum_{k=1}^L \lambda_k^2 A_k^2 = \sum_{k=1}^L \lambda_k^2 \mathbb{1} = \mathbb{1}$ and thus $|\sum_{k=1}^L \lambda_k \langle A_k \rangle| = |\langle \sum_{k=1}^L \lambda_k A_k \rangle| \leq 1$ for all states, since for any observable $\langle X \rangle^2 \leq \langle X^2 \rangle$. Interpreting the expression $\sum_{k=1}^L \lambda_k \langle A_k \rangle$ as the euclidian scalar product of the vector of coefficients λ_k and the vector of expectation values $\langle A_k \rangle$, and noting that the vector of coefficients λ_k is an arbitrary unit vector, we see that the vector $\langle A_k \rangle$ has a length less than or equal to 1. Observing $\sum_{k=1}^L \langle A_k^2 \rangle = L$, we obtain the lemma. \square

The converse implication is also true in the following sense, as was already shown in Ref. 22:

Lemma 5. *Let A_1, \dots, A_L be dichotomic anticommuting observables as above, and let a_1, \dots, a_L be real numbers with $\sum_{k=1}^L a_k^2 \leq 1$. Then there exists a quantum state ρ such that the numbers a_k are the expectation values of the observables, $a_k = \text{Tr}(A_k \rho)$.*

Proof. Consider the state $\rho = \frac{1}{d}(\mathbb{1} + \sum_{k=1}^L a_k A_k)$, where d is the dimension of the Hilbert space. Because of the properties of the observables, $\text{Tr}(A_k A_\ell) = d\delta_{k\ell}$. Furthermore, the observables A_k are traceless: $\text{Tr}(A_k) = \text{Tr}(A_k A_\ell A_\ell) = \text{Tr}(A_\ell A_k A_\ell) = -\text{Tr}(A_k A_\ell A_\ell) = -\text{Tr}(A_k)$. This shows that the state ρ has the desired expectation values. It remains to show that $\rho \geq 0$. But in the proof of the previous lemma, we have already seen that $|\sum_{k=1}^L a_k \langle A_k \rangle| \leq 1$. \square

The meta-uncertainty relation is thus the best possible bound on the expectation values of the observables. Note that in the case of one qubit and the three Pauli matrices, it reduces to the Bloch sphere picture. The relation has also been used to study monogamy relations for Bell inequalities.³³ Generalizing Wehner and Winter's result for the Shannon entropy,²² we can derive entropic uncertainty relations for various entropies from it.

Let A be an observable with eigenvalues ± 1 and $x = [\text{Tr}(A\rho)]^2$ its squared expectation value. Then the probability distribution for the measurement outcomes of A is given by $P = (\frac{1+\sqrt{x}}{2}, \frac{1-\sqrt{x}}{2})$ or $P = (\frac{1-\sqrt{x}}{2}, \frac{1+\sqrt{x}}{2})$. Any entropy S , being invariant under permutation of P , is thus a function of x , which we denote by \tilde{S} ,

$$\tilde{S}(x) = S(A|\rho) = S\left(\left(\frac{1 \pm \sqrt{x}}{2}, \frac{1 \mp \sqrt{x}}{2}\right)\right). \quad (11)$$

We say that the entropy S is concave in the squared expectation value if the function \tilde{S} is concave. This property is the crucial condition for the following entropic uncertainty relation. We shall also assume that for the peaked probability distribution, the entropy has the value zero, $\tilde{S}(1) = 0$.

Theorem 6. *Let A_1, \dots, A_L be observables which anticommute pairwise $\{A_k, A_\ell\} = 0$ for $k \neq \ell$ and which have eigenvalues ± 1 and let S be an entropy which is concave in the squared expectation value (that is, an entropy for which the function \tilde{S} defined in Eq. (11) is concave). Then*

$$\min_{\rho} \frac{1}{L} \sum_{k=1}^L S(A_k|\rho) = \frac{L-1}{L} S_0, \quad (12)$$

where $S_0 = S((\frac{1}{2}, \frac{1}{2}))$ is the entropy value of the flat probability distribution.

Proof. For the case of the Shannon entropy the proof was given in Ref. 22. Let $x_k = [\text{Tr}(A_k \rho)]^2$. Lemma 4 states that \vec{x} lies in the simplex defined by $\sum_{k=1}^L x_k \leq 1$ and $x_k \geq 0$. As the function \tilde{S} is concave on the interval $[0, 1]$, the function $\vec{x} \mapsto \sum_k \tilde{S}(x_k)$ is concave on the simplex. Thus it attains its minimum at an extremal point of the simplex, that is, $x_k = 1$ for one k and $x_\ell = 0$ for $\ell \neq k$. At an extremal point, $1/L \sum_{k=1}^L \tilde{S}(x_k) = S_0(L-1)/L$. \square

Before commenting on the implications of this theorem, we discuss which entropies satisfy the requirement of being concave in the squared expectation value. For the Shannon entropy, this was already shown in Ref. 22. The Tsallis entropy can be treated analogously, though one has to distinguish between different parameter ranges.

Lemma 7. *The Tsallis entropy S_q^T of a dichotomic observable is concave in the squared expectation value (that is, the function \tilde{S}_q^T defined as in Eq. (11) is concave on the interval $[0, 1]$) for parameter values $1 < q < 2$ and $3 < q$, but convex for $2 < q < 3$.*

Proof. Explicitly,

$$\tilde{S}_q^T(x) = \frac{1}{q-1} \left[1 - \left(\frac{1+\sqrt{x}}{2} \right)^q - \left(\frac{1-\sqrt{x}}{2} \right)^q \right]. \quad (13)$$

For $q = 2$ and $q = 3$, this function is easily seen to be linear. For the second derivative, we obtain

$$\partial_x^2 \tilde{S}_q^T(x) = \frac{q}{q-1} \frac{1}{2^{q+2}} \frac{1}{x^{3/2}} \left\{ (1+\sqrt{x})^{q-2} [1-\sqrt{x}(q-2)] - (1-\sqrt{x})^{q-2} [1+\sqrt{x}(q-2)] \right\}. \quad (14)$$

Substituting $y = \sqrt{x}$ and omitting the prefactor (which is always positive), we arrive at the function

$$f_q(y) = (1+y)^{q-2} [1-y(q-2)] - (1-y)^{q-2} [1+y(q-2)]. \quad (15)$$

Observing that $f_q(0) = 0$, we note that $f_q(y)$ is positive (negative) for all $0 < y \leq 1$ if its derivative $f'_q(y)$ is positive (negative) for all $0 < y \leq 1$. The derivative is given by

$$f'_q(y) = -(q-2)(q-1)y[(1+y)^{q-3} - (1-y)^{q-3}]. \quad (16)$$

For $1 < q < 2$, the prefactor $-(q-2)(q-1)$ is positive and the term in the square brackets is negative; for $2 < q < 3$, the prefactor is negative and the term in the brackets is still negative; for $q > 3$, the prefactor is negative and the term in the brackets positive. This proves the lemma. \square

Let us add some remarks on the theorem. The Shannon entropy has the required concavity in the squared expectation value, and the resulting uncertainty relation is the one found by Wehner and Winter.²² For the Tsallis entropy S_q^T , we have to distinguish between different parameter ranges: for parameter values $q = 2$ and $q = 3$ this entropy is, up to a constant factor, equal to the variance, $S_2^T(A|\rho) = 1/2\Delta^2(A)$ and $S_3^T(A|\rho) = 3/8\Delta^2(A)$, and the uncertainty relation is equivalent to the meta-uncertainty relation itself. Thus it is the optimal uncertainty relation for these observables. The relation based on the Shannon entropy is strictly weaker.

In Lemma 7, we have shown that the Tsallis entropy satisfies the condition of Theorem 6 for parameter values $1 < q \leq 2$ and $3 \leq q$. The entropy value for the flat probability distribution, which determines the bound, is $S_0 = (1 - 2^{1-q})/(q-1)$. In the special case of the observables σ_x and σ_y and parameter $q \in [2n-1, 2n]$ with $n \in \mathbb{N}$, the uncertainty relation was derived before in Footnote 32 of Ref. 9.

As we remarked above, Lemmas 4 and 5 provide a complete characterization of the set of expectation values of dichotomic anticommuting observables which can originate from valid quantum states. Deriving uncertainty relations from them means approximating this set from the outside. This is illustrated in Fig. 2.

In the parameter range $2 < q < 3$, the Tsallis entropy does not satisfy the condition for the theorem (see

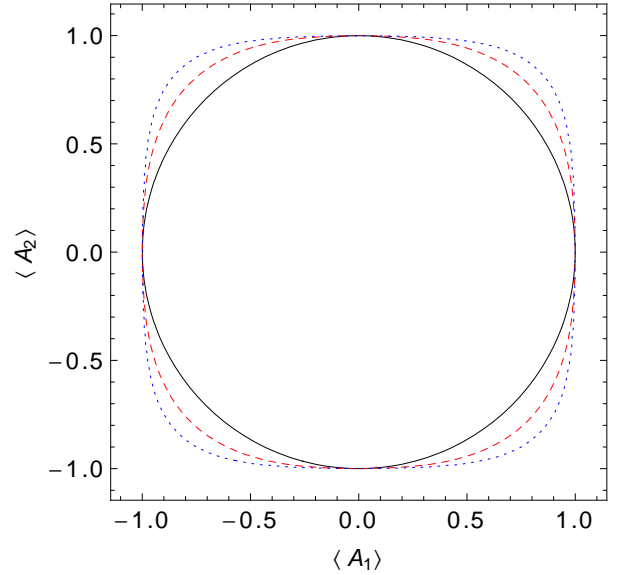


FIG. 2. Bounds on the expectation values $(\langle A_1 \rangle, \langle A_2 \rangle)$ for two dichotomic anticommuting observables provided by different uncertainty relations. The black solid line corresponds to the meta-uncertainty relation Lemma 4, which can also be understood as an entropic uncertainty relation for the Tsallis entropy with parameter value $q = 2$ or $q = 3$. The red dashed line and the blue dotted line correspond to the entropic uncertainty relation Theorem 6 for the Shannon entropy and the Tsallis entropy with $q = 8$, respectively.

Lemma 7). An exceptional behaviour of the Tsallis entropy in this parameter range was also reported in Ref. 9. The collision entropy or Rényi entropy of order 2 and the min-entropy do not satisfy the condition either. The uncertainty relations for these entropies given in Ref. 22 also follow from the meta-uncertainty relation, but do not fit into this scheme.

In the following section, we will apply Theorem 6 to the stabilizing operators of two stabilizer states.

Uncertainty relations for several observables can also be constructed without requiring anticommutativity: applying a result by Mandayam *et al.*³² to a set of stabilizer bases $\mathcal{A}_1, \dots, \mathcal{A}_L$ with basis vectors denoted by $\mathcal{A}_k = \{|a_i^{(k)}\rangle\}_i$, one finds for their min-entropies

$$\frac{1}{L} \sum_{k=1}^L S^{\min}(\mathcal{A}_k|\rho) \geq -\log \left[\frac{1+r(L-1)}{L} \right], \quad (17)$$

where

$$r = \max_{k \neq \ell} \max_{i,j} |\langle a_i^{(k)} | a_j^{(\ell)} \rangle| \quad (18)$$

is the maximal overlap of the basis states. We omit the proof, since this relation readily follows from Appendix C.1 of Ref. 32.

V. AN UNCERTAINTY RELATION FOR STABILIZING OPERATORS

In this section, we will apply the uncertainty relation for anticommuting dichotomic observables (Theorem 6) to the elements of a pair of stabilizer groups.

Let S and T be two n -qubit stabilizer groups. Throughout this section, we consider two operators as equal if they differ only by a minus sign. Define $\tilde{S} = S \setminus T$ and $\tilde{T} = T \setminus S$. Let $M = \tilde{S} \cup \tilde{T}$ be the symmetric difference of S and T and denote its elements by A_1, \dots, A_L . In Theorem 10, we will give a lower bound on $1/L \sum_{k=1}^L S(A_k|\rho)$.

We begin by proving the following lemma.

Lemma 8. *Let S be a stabilizer group and g a Pauli operator (that is, a tensor product of Pauli matrices and the identity matrix) which anticommutes with an element $s_0 \in S$. Then g anticommutes with exactly half of the elements of S .*

Proof. Choose any $s_1 \in S$ with $s_1 \neq s_0$ and let $s_2 = s_0 s_1$. We will now show that g anticommutes with s_2 if it commutes with s_1 and vice versa. Consider first the case $[s_1, g] = 0$. The identity $\{AB, C\} = A[B, C] + \{A, C\}B$ shows that $\{s_2, g\} = \{s_0 s_1, g\} = s_0[s_1, g] + \{s_0, g\}s_1 = 0$. Consider now the case $\{s_1, g\} = 0$. Using the same identity, we obtain $s_0[s_2, g] = \{s_0 s_2, g\} - \{s_0, g\}s_2 = \{s_1, g\} - \{s_0, g\}s_2 = 0$ and thus $[s_2, g] = 0$. We now iterate this procedure by choosing $s_3 \in S \setminus \{s_0, s_1, s_2\}$ and using it in place of s_1 . (Note that $s_0 s_3 \neq s_1$ and $\neq s_2$.) The operator s_0 is kept fixed during the whole iteration. In this way, S can be divided into pairs of observables, each consisting of one element commuting with g and one anticommuting with g . Note that s_0 forms a pair with the identity. \square

Excluding the trivial case $S = T$, any element of T anticommutes with at least one element of S , since at most 2^n orthogonal (with respect to the Hilbert–Schmidt scalar product) unitary $2^n \times 2^n$ -matrices can commute pairwise (see, e.g., Ref. 31). Thus $L = |M|$ varies from 2^n to $2(2^n - 1)$.

We will now see, using only combinatorial reasoning, that this lemma implies that M can be divided into anticommuting pairs. We shall need the following combinatorial result:

Marriage theorem. *Consider a bipartite graph, that is, two disjoint sets of vertices U and V and a collection of edges, each connecting a vertex in U with a vertex in V . We consider the case of $|U| = |V|$. Then the graph contains a perfect matching, that is, the vertices can be divided into disjoint pairs of connected vertices, if and only if the following marriage condition is fulfilled: For each subset U' of U , the set V' of vertices in V connected to vertices in U' is at least as large as U' .*

This theorem was first proven in Ref. 34.

Lemma 9. *The symmetric difference M of any two stabilizer groups S and T can be divided into anticommuting pairs of operators.*

Proof. We show that the marriage condition is fulfilled. Let S' be any subset of \tilde{S} . Consider first the case $|S'| > 2^{n-1}$. Then any $t \in T$ anticommutes with at least one $s \in S'$, because any such t anticommutes with exactly 2^{n-1} elements of \tilde{S} . Thus the number of $t \in \tilde{T}$ anticommuting with at least one $s \in S'$ is $|\tilde{T}| = |\tilde{S}| \geq |S'|$. Consider now the case $|S'| \leq 2^{n-1}$. For any $s \in S'$, we then find 2^{n-1} elements of \tilde{T} anticommuting with s . Thus the number of $t \in \tilde{T}$ anticommuting with at least one $s \in S'$ is $2^{n-1} \geq |S'|$. \square

We are now ready to state the main result of this section.

Theorem 10. *Let $M = \{A_1, \dots, A_L\}$ be the symmetric difference of two stabilizer groups. Then*

$$\frac{1}{L} \sum_{k=1}^L S(A_k|\rho) \geq \frac{1}{2} S_0 \quad (19)$$

holds, where S is an entropy which is concave in the squared expectation value (that is, an entropy for which the function \tilde{S} defined in Eq. (11) is concave) and S_0 is the entropy value of the flat probability distribution. Any basis state of either stabilizer basis attains the lower bound.

Proof. Due to Theorem 6 the uncertainty relation $S(A_k|\rho) + S(A_\ell|\rho) \geq S_0$ holds for any anticommuting pair A_k, A_ℓ . Lemma 9 states that M consists of $L/2$ such pairs. This shows the uncertainty relation. The density matrix of the stabilizer state defined by the group $T = \{T_k\}$ is given by

$$\rho_T = \frac{1}{2^n} \sum_{k=1}^{2^n} T_k \quad (20)$$

(see, e.g., Ref. 23). Thus

$$\text{Tr}(A_k \rho_T) = \frac{1}{2^n} \sum_{\ell=1}^{2^n} \text{Tr}(A_k T_\ell) = 0 \quad \text{for all } A_k \notin T, \quad (21)$$

showing $S(A_k|\rho_T) = S_0$ for $L/2$ observables A_k . \square

This relation is not maximally strong. This is due to the fact that some of the observables commute.

VI. CONCLUSION

We demonstrated how the stabilizer formalism can be combined with the theory of entropic uncertainty relations. We focussed on two problems: the characterization of pairs of measurement bases which give rise to strong

uncertainty relations and the generalization of the theory to the many-observable setting.

Concerning the first question, we showed that for the measurement in any two stabilizer bases the Maassen–Uffink relation is tight (Theorem 2). We also demonstrated how the stabilizer formalism can be used to compute the overlap of the basis states, which gives the lower bound on the entropy sum.

Concerning the second question, we generalized a result by Wehner and Winter on the Shannon entropy for several dichotomic anticommuting observables to a larger class of entropies (Theorem 6). Comparing the strengths of these uncertainty relation, we saw that entropic relations are not necessarily stronger than variance-based ones. Indeed, in the case of dichotomic anticommuting observables the variance-based uncertainty relation is optimal in the sense that it exactly describes the set of expectation values which can originate from valid quantum states. As an application of Theorem 6, we derived an uncertainty relation for the elements of two stabilizer groups (Theorem 10).

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Appendix A: Alternative proof of Theorem 2

In this appendix, we give an alternative proof of Theorem 2, which is not based on any previous results on mutually unbiased bases. Let $S = \{S_k\}$ and $T = \{T_k\}$ be two n -qubit stabilizer groups with stabilizer states $|S\rangle$ and $|T\rangle$. Define $S^+ = S \cap T$, where, unlike in the first proof, we consider two operators as distinct if they differ by a minus sign. Also define $S^- = S \cap -T$. Both S^+ and $S^+ \cup S^-$ are easily seen to be subgroups of S . By Lagrange’s theorem $|S^+| = 2^p$ and $|S^+ \cup S^-| = 2^q$ with some $p \in \{1, 2, \dots, n\}$ and $q \in \{p, p+1, \dots, n\}$. The projectors onto the stabilizer states are given by

$$|S\rangle\langle S| = \frac{1}{2^n} \sum_{k=1}^{2^n} S_k \quad (\text{A1})$$

and similarly for $|T\rangle$ (see, e. g., Ref. 23). Thus

$$\begin{aligned} |\langle S|T\rangle|^2 &= \frac{1}{2^{2n}} \sum_{k,\ell=1}^{2^n} \text{Tr}(S_k T_\ell) \\ &= \frac{1}{2^n} (|S^+| - |S^-|) \\ &= \frac{1}{2^n} (2^{p+1} - 2^q) \\ &= \begin{cases} 2^{q-n} & \text{for } p = q, \\ 0 & \text{for } p = q - 1. \end{cases} \end{aligned} \quad (\text{A2})$$

The case $p < q - 1$ cannot occur since it would give a negative value of $|\langle S|T\rangle|^2$ and thus lead to a contradiction.

Consider now another state $|T'\rangle$ of the stabilizer basis of T . This state is again a stabilizer state, whose stabilizing operators are equal to those of $|T\rangle$ up to some minus signs. In particular $S^+ \cup S^-$ and thus q are the same for $|T\rangle$ and for $|T'\rangle$, and Lemma 1 applies. \square

Appendix B: A recurrence relation for the overlap of two graph state bases

In this appendix, we derive a recurrence relation for the scalar products $\langle G_i^{(1)} | G_j^{(2)} \rangle$ of two graph state bases $|G_i^{(1)}\rangle$ and $|G_j^{(2)}\rangle$ and use it to determine the Maassen–Uffink bound for certain classes of graph states. Recall that any of these basis states is obtained from the state $|+\rangle^{\otimes n}$ by applying first controlled phase gates $C = \text{diag}(1, 1, 1, -1)$ and then local phases σ_z . Since all these operations commute, we can move all phase gates in the scalar product $\langle G_i^{(1)} | G_j^{(2)} \rangle$ to the right and all local phases to the left. This corresponds to replacing $G^{(1)}$ by the empty or completely disconnected graph and $G^{(2)}$ by the “sum modulo 2” of the graphs $G^{(1)}$ and $G^{(2)}$. Similarly, it does not restrict the set of values of $\langle G_i^{(1)} | G_j^{(2)} \rangle$ if we consider only one state of the basis $|G_j^{(2)}\rangle$, such as the graph state $|G^{(2)}\rangle$ itself. The graph state basis of the empty graph consists of all tensor products of the eigenstates of σ_x . We can write those states as $H^{\otimes n} |\vec{y}\rangle$, where $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate and $|\vec{y}\rangle$ for $\vec{y} \in \mathbb{F}_2^n$ is a state of the standard basis in binary notation.

[As the Hadamard gate is a local Clifford operation, the state $H^{\otimes n} |G^{(2)}\rangle$ is a stabilizer state. This shows that the scalar products $\langle G_i^{(1)} | G^{(2)} \rangle$ can be understood as the coefficients of a stabilizer state with respect to the standard basis. It has been shown that for any stabilizer state these coefficients are 0, ± 1 , and $\pm i$, up to a global normalization (see Theorem 5 and the paragraph below in Ref. 35. For an alternative proof, see Ref. 36.) This constitutes yet another proof of Theorem 2 for the case of graph state bases.]

As an example, consider the graphs in Fig. 1 (a) and (b). Up to local unitaries, the corresponding graph states

are the 4-qubit GHZ state and the 4-qubit linear cluster state, but the uncertainty relation is not invariant under these local unitary operations. By the above remark the Maassen–Uffink bound for the corresponding bases is equal to the bound for the empty graph and the “sum” of the graphs, which in our case is given by Fig. 1 (c). The latter is again equal to the graph (b), up to a permutation of vertices.

Let us return to the explicit calculation of the overlaps. In the standard basis, we have

$$H^{\otimes n}|\vec{y}\rangle = \frac{1}{2^{n/2}} \sum_{\vec{x} \in \mathbb{F}_2^n} (-1)^{\sum_i y_i x_i} |\vec{x}\rangle. \quad (\text{B1})$$

The adjacency matrix of an n -qubit graph is the symmetric $n \times n$ -matrix A whose entry A_{ij} is 1 if there is an edge between vertices i and j and 0 if there is not. It thus provides a complete description of the graph state, which we will therefore denote by $|G_A\rangle$. The representation of

this state in the standard basis is (see Proposition 2.14 in Ref. 37)

$$|G_A\rangle = \frac{1}{2^{n/2}} \sum_{\vec{x} \in \mathbb{F}_2^n} (-1)^{\sum_{i < j} x_i A_{ij} x_j} |\vec{x}\rangle. \quad (\text{B2})$$

The scalar products are thus given by

$$\begin{aligned} R_n(\vec{y}, A) &:= \langle \vec{y} | H^{\otimes n} | G_A \rangle \\ &= \frac{1}{2^n} \sum_{\vec{x} \in \mathbb{F}_2^n} (-1)^{\sum_i y_i x_i + \sum_{i < j} x_i A_{ij} x_j}. \end{aligned} \quad (\text{B3})$$

To derive a recurrence relation, we write

$$\vec{x} = \begin{pmatrix} \xi \\ \vec{x}' \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} v \\ \vec{y}' \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \vec{a}'^t \\ \vec{a}' & A' \end{pmatrix} \quad (\text{B4})$$

and obtain

$$\begin{aligned} R_n(\vec{y}, A) &= \frac{1}{2^n} \sum_{\xi \in \{0,1\}} \sum_{\vec{x}' \in \mathbb{F}_2^{n-1}} (-1)^{v\xi + \sum_i y'_i x'_i + \xi \sum_i a'_i x'_i + \sum_{i < j} x'_i A'_{ij} x'_j} \\ &= \frac{1}{2^n} \sum_{\vec{x}' \in \mathbb{F}_2^{n-1}} (-1)^{\sum_i y'_i x'_i + \sum_{i < j} x'_i A'_{ij} x'_j} + \frac{1}{2^n} (-1)^v \sum_{\vec{x}' \in \mathbb{F}_2^{n-1}} (-1)^{\sum_i (y'_i + a'_i) x'_i + \sum_{i < j} x'_i A'_{ij} x'_j} \\ &= \frac{1}{2} R_{n-1}(\vec{y}', A') + (-1)^v \frac{1}{2} R_{n-1}(\vec{y}' + \vec{a}', A'). \end{aligned} \quad (\text{B5})$$

This is the desired recurrence relation.

We already know that $R_{n-1}(\vec{y}', A') \in \{0, \pm r_{A'}, \pm i r_{A'}\}$ for all \vec{y}' for some $r_{A'}$. Since the coefficients R are real, $R_{n-1}(\vec{y}', A') \in \{0, \pm r_{A'}\}$. Thus we have $R_n(\vec{y}, A) \in \{0, \pm \frac{1}{2} r_{A'}, \pm r_{A'}\}$. This shows that either $r_A = \frac{1}{2} r_{A'}$ or $r_A = r_{A'}$, depending on \vec{a}' .

The Maassen–Uffink relation for the graph state bases is maximally strong if and only if $r = 2^{-n/2}$. On the other hand, r is an integer multiple of 2^{-n} (even 2^{1-n}). One can see this by induction with the recurrence relation. This shows that this uncertainty relation is never maximally strong for graph state bases with an odd number of qubits.

We will now use the recurrence relation to compute r for certain classes of states, and by doing so, construct maximally strong uncertainty relations for all even numbers of qubits. We assume one graph to be empty and vary only the other one. The application of the recurrence relation is particularly easy if $R_{n-1}(\vec{y}', A')$ is independent of \vec{y}' , up to a sign.

First we show by induction that for the fully connected graph with an even number n of qubits, we have $R_n(\vec{y}, A) = \pm 2^{-n/2}$ for all \vec{y} . For $n = 2$, we have $r_2 = 1/2$. Assume the assertion to be true for $n - 2$. Let $A, A',$ and A'' be the fully connected adjacency matrices for $n, n - 1,$ and $n - 2$ qubits, respectively, and

$\vec{a}' = (1, 1, \dots, 1) \in \mathbb{F}_2^{n-1}$, $\vec{a}'' = (1, 1, \dots, 1) \in \mathbb{F}_2^{n-2}$ and similarly for $\vec{0}$. Then

$$\begin{aligned} R_n(\vec{0}, A) &= \frac{1}{2} R_{n-1}(\vec{0}', A') + \frac{1}{2} R_{n-1}(\vec{a}', A') \\ &= \frac{1}{2} \left[\frac{1}{2} R_{n-2}(\vec{0}'', A'') + \frac{1}{2} R_{n-2}(\vec{a}'', A'') \right] \\ &\quad + \frac{1}{2} \left[\frac{1}{2} R_{n-2}(\vec{a}'', A'') - \frac{1}{2} R_{n-2}(\vec{0}'', A'') \right] \\ &= \frac{1}{2} R_{n-2}(\vec{a}'', A'') \\ &= \pm \frac{1}{2} \frac{1}{2^{(n-2)/2}} = \pm \frac{1}{2^{n/2}}. \end{aligned} \quad (\text{B6})$$

This implies that $R_n(\vec{y}, A) \in \{0, \pm 2^{-n/2}\}$ for all \vec{y} . But because of normalization, $R_n(\vec{y}, A) = 0$ is not possible. This shows the assertion. For the fully connected graph with an odd number of qubits, we have

$$\begin{aligned} R_n(\vec{y}, A) &= \frac{1}{2} R_{n-1}(\vec{y}', A') \pm \frac{1}{2} R_{n-1}(\vec{y}' + \vec{a}', A') \\ &\in \left\{ 0, \pm \frac{1}{2^{(n-1)/2}} \right\}. \end{aligned} \quad (\text{B7})$$

The generalization of the state in Fig. 1 (b), which is equivalent under local unitary operations to the linear

cluster state, can be treated in exactly the same way, and we obtain the same results for r_n .

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